1 Introduction

If we are interested in the truth conditional semantics of natural language, our theories should say something about syntactically well formed expressions which appear to be semantically infelicitous. This paper is concerned with two such examples of infelicity, both of which involve false existence presuppositions, one with a non-denoting definite descriptor, the other with a non-denoting anaphor:

The present queen of France is bald.
Every man walked in. He whistled.

Both of these examples seem to presuppose the existence of terms which do not exist. Even the negation of these discourses require the existence of these terms. In general, the problem to be addressed here is the status of such examples in truth conditional semantics.

One possibility is to use a notion of truth-value gap (van Fraassen, 1966), which would allow us to say that propositions representing sentences, and discourse, can be neither true nor false. This raises the question of whether the truth-value gaps should be something expressed in the representation, or something about the representation and its models. In the former case, we might adopt a non-classical three-valued logic. In the latter case, we assume that the logic is incomplete with respect to certain assertions. In both cases, we seem to be saying that (the representation of) infelicitous sentences are the kinds of objects to which truth values can be ascribed.

However, there is an alternative that appears in some theories which address the logical paradoxes, such as the Liar: “this sentence is false”. This is a well formed sentence, yet any classical theory which gives it a truth value becomes inconsistent.

Many formalisms brush this problem under the carpet, and use strong typing in order to prevent the self-application present in such examples. Exception to this are theories of truth, propositions and properties (often referred to, collectively, as “Property Theory”). Such theories take seriously the idea that propositions should be considered to be primitive objects, whose truth conditions can be expressed by means of a truth predicate.

In general, the presence of a truth predicate gives rise to logical paradoxes in formal theories. Property Theory can avoid these paradoxes by effectively limiting the application of the truth predicate to only non-problematic terms. In Turner’s Property Theory (PT) which axiomatises Aczel’s Frege Structures (Turner, 1982; Aczel, 1980), the axioms of truth governing the truth predicate can only be applied to those terms which we can prove to be “propositions”. The axioms governing propositionhood do not allow a proof that paradoxical expressions are propositions, despite their appearance.

The paradoxes are thus taken to be examples of a “category mistake”: although they have the form of a proposition, it is a mistake to assume that their semantic representation can be classified as such.

It seems to me that the same argument can be applied in the case of sentences with false existence presuppositions: although they have the appropriate form, we might argue that it is a category mistake to assume that their semantic representation must be taken to be a proposition. Felicity is thus equated with propositionhood.

2 Property Theory

2.1 The Basic Theory

Conceptually, Turner’s property theory PT can be split into two components, or levels. The first is a language of terms, which consists of the untyped λ-calculus, embellished with logical constants. A restricted class of these terms will correspond to propositions. When combined appropriately using the logical constants, other proposi-
tions result. As an example, given the propositions \( t, s \), the ‘conjunction’ of these, \( t \land s \), is also a proposition, where \( \land \) is a logical constant.

Some of the propositions will, further, be true propositions. When combining propositions with the logical constants, the truth of the resultant proposition will depend upon the truth of the constituent propositions. Considering the previous example, if \( t, s \) are both propositions, then \( t \land s \) will be a true proposition if and only if \( t \) and \( s \) are true propositions.

There may be terms that form propositions when applied to another term. These terms are the properties. The notion of logical equivalence: propositions are not to depend on the truth of the constituent propositions. The Language of terms

This is a highly intensional theory as the notion of equality is that of the \( \lambda \)-calculus, which is weaker than the notion of logical equivalence: propositions are not to be equated just because they are always true together; similarly, properties are not to be equated just because they hold of the same terms.

### 2.1.1 The Formal Theory

The following presents a formalisation of the languages of terms and wff, together with the axioms that provide the closure conditions for \( P \) and \( T \).

#### The Language of Terms

**Basic Vocabulary:**

- Individual variables: \( x, y, z, \ldots \)
- Individual constants: \( c, d, e, \ldots \)
- Logical constants: \( \forall, \land, \lor, \Rightarrow, \exists, \Theta \)

**Inductive Definition of Terms:**

(i) Every variable or constant is a term.

(ii) If \( t \) is a term and \( x \) is a variable then \( \lambda x.t \) is a term.

(iii) If \( t \) and \( t' \) are terms then \( t(t') \) is a term.

#### The Language of Wff

**Inductive Definition of Wff:**

(i) If \( t \) and \( s \) are terms then \( s = t, P(t), T(t) \) are atomic wff.

(ii) If \( \varphi \) and \( \varphi' \) are wff then \( \varphi \land \varphi', \varphi \lor \varphi', \varphi \rightarrow \varphi', \sim \varphi \) are wff.

(iii) If \( \varphi \) is a wff and \( x \) a variable then \( \exists x \varphi \) and \( \forall x \varphi \) are wff.

The theory is governed by the following axioms:

**Axioms of the \( \lambda \beta \)-calculus**

\[
\lambda x. t = \lambda y. t[y/x] \quad y \text{ not free in } t
\]

\[
(\lambda x. t)t' = [t'/x]
\]

This defines the equivalence of terms.

The closure conditions for propositionhood are given by the following axioms:

**Axioms of Propositions**

(i) \( P(t) \& P(s) \rightarrow P(t \land s) \)

(ii) \( P(t) \& P(s) \rightarrow P(t \lor s) \)

(iii) \( P(t) \& (T(t) \rightarrow P(s)) \rightarrow P(t \Rightarrow s) \)

(iv) \( P(t) \rightarrow P(\neg t) \)

(v) \( \forall x P(t) \rightarrow P(\Theta \lambda x.t) \)

(vi) \( \forall x P(t) \rightarrow P(\Xi \lambda x.t) \)

(vii) \( P(s \approx t) \)

Truth conditions can be given for those terms that are propositions:

**Axioms of Truth**

(i) \( P(t) \& P(s) \rightarrow (T(t \land s) \leftrightarrow T(t) \& T(s)) \)

(ii) \( P(t) \& P(s) \rightarrow (T(t \lor s) \leftrightarrow T(t) \lor T(s)) \)

(iii) \( P(t) \& (T(t) \rightarrow P(s)) \rightarrow (T(t \Rightarrow s) \leftrightarrow T(t) \rightarrow T(s)) \)

(iv) \( P(t) \rightarrow (T(\neg t) \leftrightarrow \sim T(t)) \)

(v) \( \forall x P(t) \rightarrow (T(\Theta \lambda x.t) \leftrightarrow \forall x T(t)) \)

(vi) \( \forall x P(t) \rightarrow (T(\Xi \lambda x.t) \leftrightarrow \exists x T(t)) \)

(vii) \( T(t \approx s) \leftrightarrow t = s \)

(viii) \( T(t) \rightarrow P(t) \)

The last axiom states that only propositions may have truth conditions.

Note that the quantified propositions \( \Theta \lambda x.t, \Xi \lambda x.t \) can be written as \( \Theta x(t), \Xi x(t) \), where the \( \lambda \)-abstraction is implicit.

This basic theory is very weak. The general approach for analysing semantic phenomena is to amend the theory either definitionally, or by strengthening it with more axioms and primitive notions. This is, of course, in addition to obtaining appropriate representations for natural language phrases.

It is straightforward to define some simple types. The notions of \( n \)-place relations can be defined recursively:

(i) \( \text{Rel}_0(t) \leftrightarrow P(t) \)

(ii) \( \text{Rel}_n(\lambda x.t) \leftrightarrow \text{Rel}_{n-1}(t) \)

As can be seen, types in this theory correspond with predicates. We can write \( \text{Rel}_1(t) \) as \( \text{Pty}(t) \).

Types can be declared that correspond with the notions of quantifier and determiner Montague’s IL (Turner, 1992). Functional types can be defined as
follows: if \( P, Q \) are types, then the functional type
\[
(Q \rightarrow R) \rightarrow S
\]
can be defined with:

\[
(Q \rightarrow R)(g) = \forall x (Q(x) \rightarrow R(gx))
\]

3 Definite Descriptors in Property Theory

In PT, not all expressions represent propositions. This is an essential aspect of its treatment of paradoxical terms, and as mentioned above, it may be used to model felicity in discourse. We can set up the axioms in such a way that sentences whose presuppositions are not met cannot be proven to be propositions. This can be illustrated with definite descriptors. We can define a class of natural language denotable individuals, and a class of natural language denotable properties. Given \( p \) in \( \text{Pty}_\Delta \) and \( s \) in \( \Delta \), then we can prove that \( ps \) is a proposition by adding the following axiom:

\[
(\text{Pty}_\Delta p \& \Delta s) \rightarrow P(ps)
\]

We may require that a definite descriptor \( \sigma x q x \) (the \( x \) such that \( T(qx) \)) is only provably in \( \Delta \) if \( q \) has an extension in \( \Delta \):

\[
\exists y (\Delta y \& T(qy)) \rightarrow \Delta(\sigma x q x)
\]

Thus, when it comes to evaluating sentences such as “the present queen of France is bald”, we cannot prove that the representation of the sentence is a proposition. This is because we cannot prove that “the present queen of France” is a natural language denotable, as there is no “present queen of France”. The failure to prove the proposition-hood of a sentence means that we cannot apply the axioms of T to determine the truth conditions of the sentence; the sentence is not the sort of object whose truth conditions should be considered.

This can be generalised to plurals and mass terms, by replacing this axiom with:

\[
\forall x (T(qx) \rightarrow \Delta x) \& \exists y (\Delta y \& T(qy)) \rightarrow \Delta(\sigma x q x)
\]

which says that a definite descriptor is denotable if the associated property is a property of denotables, and there is a denotable in its extension. This form of this axiom is justified in (Fox, 1993).

Notice that with definite descriptors, what is referred to as ‘accommodation’ could be modelled by some form of abdication in the representation. Also, the axioms for P and T with material implication cope with the so-called projection problem for conditionals: existence presuppositions arising from the consequent are not projected to the top level if they are satisfied by the truth of the antecedent. To cope with the projection problem with disjunction requires some alterations to the axioms (Fox, 1993).

4 Anaphora in Type Theory

A similar argument can be given for pronouns which lack an antecedent. Indeed, this is embodied in the constructive-type theoretic approach to natural language semantics (Ranta, 1991; Davila-Perez, 1994).

In summary, they take sentences in a discourse to provide types or specifications of objects which satisfy them. We might see parallels with Discourse Representation Theory (DRT) (Kamp, 1981) here, where natural language provides type specifications (the conditions) for discourse objects. In DRT, we might say that a discourse is true if we can anchor the discourse markers in a way which satisfies the conditions. The constructive-type approach is different in that objects (proofs or witnesses) must also be produced for the ‘conditions’ themselves.

In effect, objects (witnesses) which satisfy a discourse are passed on to the next sentence as an argument. These witnesses are either objects structured as pairs (which arise from indefinites, relative clauses, and sequences of sentences) or function types, which are specified by the representation of universal quantification and “if then” constructions.

Singular pronouns are resolved by representing them with selector functions, which can pick elements from nested pairs.

Universal quantification is usually taken to block singular anaphoric reference, as in the example: “Every man walked in. He whistled.”. In DRT this is achieved by ‘box-splitting’. In Dynamic Predicate Logic (Greeenendijk and Stokhof, 1991) it is achieved by stipulating that universal quantification is ‘externally static’. In the constructive type interpretation, it is blocked because the first sentence gives rise to a function, not a nested pair. Hence, selector functions cannot be applied to this without producing an ill-formed expression.

The type theory typically used for anaphora in the semantics of natural language is Martin-Löf’s Type Theory (MLTT) (Martin-Löf, 1982; Martin-Löf, 1984) (Sundholm, 1989; Ranta, 1991; Davila-Perez, 1994).

In MLTT, there are types and elements (or witnesses) of types. If \( T \) is a type, then:

\[ w : T \]

says that \( w \) is an element of that type.

Types can be seen as specifications, and elements of the types are then programs that meet the specification. Alternatively, types can be said to correspond

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with pronouns, and elements of a type with proofs of that proposition. This is a form of intuitionistic logic. A proposition is true if we can produce a proof of it. If the class of specified proofs is non-empty (or inhabited), then the proposition is true.

Various sorts of complex types can be defined in MLTT. Those of interest for Donkey sentences and intersentential anaphora are the dependent types. These allow us to treat the witness for a type as a context for subsequent types. Taking type operator $\Sigma$ as a relevant example:

\[ \Sigma f \rightarrow g \]

is a type formed from the types $f$, $g$. Viewed as a proposition, it is true if we can find an $h$ such that:

\[ h : (\Sigma f \rightarrow g) \]

Witnesses for this type are pairs, so $h = \langle a, b \rangle$. The definition of types of this form in MLTT requires that if:

\[ \langle a, b \rangle : (\Sigma f \rightarrow g) \]

then it must be the case that:

\[ a : f \]
\[ b : g[a/x] \]

That is, $h$ consists of a proof of $f$ and a proof of $g$, where the proof of $f$ has been substituted for all free occurrences of $x$ in $g$. Here we see the context dependence created by the $\Sigma$ type: the proof required for $g$ depends upon the proof give for $f$. If the proof of $f$ in effect contains discourse referents, then these become available in $g$.

In natural language semantics, the $\Sigma$ operator can be used to represent indefinites, sentential conjunction, and relative clauses. Using $\Sigma$ for indefinites means that witnesses to nominals (discourse referents in DRT) are made available to later parts of the sentence. Using $\Sigma$ for sentential conjunction means that these discourse referents are also available to subsequent sentences.

Sequences of the selector functions $\mathrm{fst}$, $\mathrm{snd}$ can be used to recurse through the nested pairs that result from discourse. The act of resolving the reference of a singular pronoun is achieved by replacing the occurrence of the pronoun in the semantics by selector functions operating on an appropriate argument.

The $\Pi$ operator is used for universal quantification and conditionals. As with box-splitting in DRT, $\Pi$ blocks singular anaphoric reference.

Skipping the details of the translation procedure, the discourse:

A man walked in. He whistled.

would be represented:

\[ \Pi y z (\Sigma x \cdot \text{man'}(x) \cdot \text{walked-in'}(y))(\text{whistled'}(\text{he}_0)) \]

This MLTT proposition will be true if there it has a witness of the form:

\[ \langle \langle m, \varphi \rangle, \psi \rangle \]

where $m$ is a man, and $\varphi$ is a proof that that man walked in, and $\psi$ is a proof that "He" whistled. More specifically, following the rules for $\Sigma$, $\varphi$ is a proof of $(\text{whistled'}(\text{he}_0))$ when all occurrences of $y$ are replaced by $\langle m, \varphi \rangle$. We can make the type specify that it was the previously mentioned man that whistled by replacing the pronoun with $\mathrm{fit}(y)$. The type expression for the consequent sentences then becomes equivalent to $\text{whistled'}(m)$. More explicitly, the sentence is true if:

\[ \langle \langle m, \varphi \rangle, \psi \rangle : \Sigma y \cdot (\Sigma x \cdot \text{man'}(x)) \cdot \text{walked-in'}(y) \cdot \text{whistled'}(\text{he}_0) \]

The singular pronoun $\text{he}_0$ is resolved by replacing with $\mathrm{fit}(y)$, so that:

\[ \psi : \text{whistled'}(\mathrm{fit}(m, \varphi)) \]
\[ = \text{whistled'}(m) \]

For universal quantification and conditionals, we need to make use of another dependent types operator $\Pi$. The expression:

\[ \Pi y z f \cdot g \]

holds of functions which take all proofs, or witnesses $w$ of $f$ to a proof of $g(w)$. This is similar to universal quantification (or implication). It can be used to represent the universal determiner "all", and "if ... then ..." constructions. So the sentence:

If a man owns a donkey, he beats it.

can be represented as:

\[ \Pi x z \cdot (\Sigma y \cdot \text{man'}(y))(\Sigma z \cdot \text{donkey'}(z))(\text{own'}(x))(\text{beats'}(\text{it}_0)(\text{he}_0)) \]

The sentence is true if we can find a function that maps objects of the form:

\[ \langle f, \langle d, \varphi \rangle \rangle \]

where $f$ is a farmer, $d$ is a donkey, and $\varphi$ is a proof that $f$ owns $d$, into a proof of:

\[ \text{beats'}(\text{it}_0, \text{he}_0) \]
The anaphora are resolved if we replace them with selectors as follows:

\[ \text{beat'}(\text{fst}(\text{snd}(x))(\text{fst}(x))) \]

where \( x \) becomes instantiated with \( (f, \langle d, \varphi \rangle) \):

\[ \text{beat'}(\text{fst}(\text{snd}(\langle f, \langle d, o \rangle \rangle))(\text{fst}(\langle f, \langle d, o \rangle \rangle))) \]

which is equivalent to:

\[ \text{beat'}df \]

For a compositional analysis, this is not the whole story. The interested reader is referred to the expositions of MLTT in natural language semantics given by Ranta and Davila-Perez for more details (Ranta. 1991; Davila-Perez. 1994).

In MLTT, our infelicitous example:

Every man walked in. He whistled.

is represented by something like:

\[ \Sigma y \exists (\forall x \exists \text{manWalkeredIn}(x)) \text{(whistled}(\text{he}_0)) \]

The pronoun in the representation of the second sentence cannot be resolved by replacing it with a selector as the previous sentence supplies a function, not a nested pair. The formation rules in MLTT do not support this representation as it would embody a type mismatch. As can be seen, this approach to discourse anaphora has parallels with the treatment of non-denoting definite descriptors given above.

Perhaps some objections to the use of MLTT for NL semantics should be noted. Firstly, it is a non-classical theory. Witnesses must be provided for both properties and propositions. In this sense, the theory conflates the notions of property and proposition.

Two further points against this theory are that: there is no means of distinguishing false propositions; and there is no extensional-intensional contrast in the basic theory. Both are required in natural language semantics.

To remedy these last two defects, and to elucidate the connection with the treatment of non-denoting definite descriptors I propose, we can implement MLTT in Property Theory, as done explicitly by Turner and Smith (Turner. 1992; Smith. 1984). Indeed, working with MLTT defined in Property Theory gives some additional flexibility, and provides a classical notion of propositions and the standard connectives.

In principle, this gives us the tools needed to give an analysis of anaphora with dependent types whilst keeping the classical distinction between properties and propositions. There is insufficient space to explore this possibility here.

5 Type Theory in Property Theory

Types can be defined in the theory as properties. We can give definitions for various operations on these types, such as intersection \( \cap \), union \( \cup \), difference \( - \), cartesian product \( \times \), disjoint union \( \oplus \), and function space \( \to \) operators (Turner. 1992). Only intersection and disjoint union will be illustrated here:

\[
\begin{align*}
\cap &=_{\text{def}} \lambda f. \lambda g. \{ x : fx \cap gx \} \\
\oplus &=_{\text{def}} \lambda f. \lambda g. \{ z : (\text{fst}(z) \approx 0 \land f(\text{snd}(z))) \lor (\text{fst}(z) \approx 1 \land g(\text{snd}(z))) \}
\end{align*}
\]

where \( \{x : t\} \) is syntactic sugar for \( Ax.t \) especially when \( t \) is a property.

The definitions of type operators trivially support the theorems:

\[
\begin{align*}
ze(t \cap s) &\iff zet \land zes \\
ze(t \oplus s) &\iff (\text{fst}(z) = 0 \land \text{snd}(z) = t) \\
&\lor (\text{fst}(z) = 1 \land \text{snd}(z) = s)
\end{align*}
\]

where \( zet \) is sugar for \( T(tx) \).

Pairs \( (,) \) and and \( \text{fst} \), \( \text{snd} \) have their usual definitions:

\[
\begin{align*}
\text{fst} &=_{\text{def}} \lambda p. pxy \, x \\
\text{snd} &=_{\text{def}} \lambda p. pxy \, y \\
(x, y) &=_{\text{def}} \lambda z. (z(x))(y)
\end{align*}
\]

so that:

\[
\begin{align*}
\text{fst}((x, y)) &=_{\beta} x \\
\text{snd}((x, y)) &=_{\beta} y
\end{align*}
\]

The dependent type operators \( \Pi \) and \( \Sigma \) of MLTT can be defined with:

\[
\begin{align*}
\Pi &=_{\text{def}} \lambda f. \lambda g. \{ h : \Theta x. \{ x \Rightarrow gx(hx) \} \} \\
\Sigma &=_{\text{def}} \lambda f. \lambda g. \{ h : f(\text{fst}(h)) \land g(\text{fst}(h))(\text{snd}(h)) \}
\end{align*}
\]

These definitions support the following theorems:

If \( \text{Pty}(f) \) and \( \forall x(x \varepsilon f \Rightarrow \text{Pty}(gx)) \) then:

\[
\text{Pty}(\Pi f) \\
\text{Pty}(\Sigma f)
\]

and:

\[
\begin{align*}
he\Pi f g &\iff \forall x(x \varepsilon f \Rightarrow hx \varepsilon gx) \\
he\Sigma f g &\iff \text{fst}(h) \varepsilon f \land \text{snd}(h) \varepsilon g(\text{fst}(h))
\end{align*}
\]

1. This drawback might be overcome if we add various universes to MLTT, but this seems to be to admit the primitive nature of propositions.

2. Further, property theory with pairs and selectors (as used in MLTT semantics) can be used to model some aspects of abstraction and application in ‘\( \cdot \cdot \cdot \text{DRT} \)’ and Aczel-Lunnon set abstraction in Situation Theory (Aczel and Lunnon. 1991).
To paraphrase these definitions, $h \Pi fg$ means that $h$ is a function which takes an element (or ‘proof’/‘witness’) of $g$ and gives a ‘proof’ of $f$ applied to that element of $f$. The expression $h \Sigma fg$ means that $h$ is a pair, where the first component of the pair is an element of $f$, and the second is an element of $g$ applied to that element of $f$.

With both of these types the evaluation of $g$ is dependent upon the chosen element, or ‘proof’, of $f$. In some sense then, the meaning of $g$ depends upon the context created by $f$. This gives us the means to give an MLTT based treatment of anaphora and discourse in PT, as illustrated before.

Now we can adapt the approach given for the definite descriptor for the anaphoric example. Essentially, if we give the basic terms used in the representation of NL appropriate types, then it is not possible to replace the pronoun with a selector function which permits us to prove that the sentence is represented by a MLTT proposition (a property in this implementation). Thus it is not possible to proof that, given a putative witness, the sentence forms a proposition.

The appropriate types are as follows: denotable individuals will be in $\Delta$, and MLTT ‘properties’ will for properties give a denotable, that is, they will be of the type $\Delta \Rightarrow \text{Pty}$.

With the infelicitous example:

“Every man walked in. He whistled.”

in the representation it is not possible to replace the pronoun with a selector that is provably a denotable (in $\Delta$), so we cannot show that the discourse is of an appropriate type to have its truth conditions considered.

As mentioned before, the proposed treatment of the first, definite descriptor example uses the classical notion of proposition, while the second, anaphoric example uses constructive propositions (sentences are represented as types). There is insufficient space here to illustrate how this might be remedied so as to achieve a uniform theory for both examples.

6 Conclusions

Using Property-theoretic semantics, it is possible to use propositionhood to characterise felicitous discourse. To consider the truth conditions of discourse with: paradoxical expressions; non-denoting definite descriptors; and pronouns without an antecedent is to make a category mistake.

It might be argued that if both the paradoxes and expressions with non-denoting nominals give rise to the absence of propositionhood, then we have conflated two very different phenomena. However, the failure to prove propositionhood arises for different reasons: with the paradoxes, it stems from failure to prove a term is a property; with existence presuppositions it comes from failure to prove that a term is denotable. In the later case, an appropriate denotable term can be added, or accommodated. Propositionhood will then follow. In the former case, no consistent assumption can be added which will allow a proof of propositionhood.

This treatment of existence presuppositions is achieved in a first-order, two-valued classical theory, with fine-grained intentionality. As such it lends support to the view that this is all that is needed for the truth conditional semantics of a large fragment of natural language.

References


