In Defence of Axiomatic Semantics

Abstract: Many semantic theories implicitly attribute a foundational status to set theory, and set-theoretic characterisations of possible worlds in particular. The goal of a semantic theory is then to find a translation of the phenomena of interest into a set-theoretic model. This can be contrasted with an axiomatic approach in which we can formulate “new” primitives and ontological categories, and devise logical rules and axioms that capture the appropriate inferential behaviour. This alternative approach might be criticised as being mere “descriptivism”, lacking predictive or explanatory power. Here we will seek to defend the axiomatic approach. Any formal account must assume some intended interpretation, but there is a sense in which such theories can provide a more honest characterisation. In contrast, the set-theoretic approach tends to conflate distinct ontological notions. Mapping a pattern of semantic behaviour into some pre-existing set-theoretic behaviour may lead to certain aspects of meaning being overlooked, or ignored.

Key words: semantics, ontology, methodology, reductionism, natural language, logic

1. Introduction

There are a number of questions we can ask about the status of logical accounts of the meaning of language. For example: When does a particular proposal count as a semantic theory? How do we judge a theory to be “correct”? What criteria can we use to decide whether one theory is “better” than another?

Here we will seek to defend what can be described as an axiomatic approach. In particular, we argue against the view (sometimes implicit, sometimes explicit) that the only credible kind of semantic analysis is one that is ultimately expressed within the language, and ontology, of set-theory.

1.1. Traditional formal semantics

A now traditional characterisation of what constitutes a formal semantic theory is one that provides a systematic translation of linguistic constructs
into an appropriate formalism that captures the salient aspects of behaviour. The systematic translation is usually “compositional” in the sense that there is (at least) one rule of translation for every rule of syntax. In the case of indicative sentences, this could be a translation of sentences into a truth-conditional logic, where the truth conditions of the translations of the sentences, and the relationships between then, accord with our intuitions about the original sentences.

This approach is predicated on the assumption that there is an appropriate formal characterisation of the syntax of the language, and that we have access to intuitions about the meaning of that language. We will not question these assumptions in the current article.

1.2. Some questions

Questions that we will consider here are: What are the “salient aspects of behaviour”, and in what sense should they be “captured”? What counts as an appropriate formalism, or interpretation? That is, what is the nature of (i) the data and (ii) the formal interpretation?

Concerning the data, we need to consider which aspects of behaviour that we are trying to capture, model or explain. Often the data are messy, with confounding aspects of behaviour. Intuitions about the data may be unclear and subject to differences of opinion.

If we wish to capture a particular aspect of behaviour, there is a question as to the most natural lines of division between it and other aspects of behaviour, and whether the appropriate lines of division are universal, or language specific. It can sometimes be unclear how to factorise the behaviour of a given example into these different aspects. There may also be questions as to whether it is right to seek to factorise behaviour in this way, or whether a more holistic approach is required.

Traditionally it has been common to focus on “toy” examples and scenarios relating to some aspect of meaning, and to capture intuitions relating to their semantic behaviour. A “fragment” of the entire language may be formalised so that the examples can be analysed in an appropriate context (Montague’s “fragments”, Montague, 1970a,b, 1973; Partee, 2001). This can help avoid making completely unjustified assumptions
about the rest of the grammar.\(^1\) With or without fragments, the idea of concentrating on toy examples might be considered a weakness. But it can be justified by appeal to natural science, where it is normal to make simplifying assumptions, and capture the behaviour of simplified systems.

Again we may question the impact of such simplifications, and whether the categories of phenomena supposedly characterised by the toy examples are in any sense “natural”, and independent. Ideally semanticists should be aware of potentially confounding linguistic data, and ought to consider cross-linguistic data as a guard against over-generalising, or “over fitting” the apparent behaviour of a small selection of examples taken from one language.

In addition to the question of the data, there is also the issue of what kinds of system are assumed appropriate as vehicles for expressing semantic behaviour. What criteria should be used to determine that one target formalisation (logic or theory) is more appropriate than another? Is it sufficient to capture the relevant logical behaviour, or are there other issues which need to be considered?

2. **Set theory and possible worlds**

Many semantic theories implicitly attribute a foundational status to set theory, and set-theoretic characterisations of possible worlds in particular.

2.1. **Examples**

It is appropriate to consider some representative examples of semantic accounts that are often given a set-theoretic characterisation.

2.1.1. **Montagovian model-theoretic semantics**

A frequently cited example of the set-theoretic approach is attributed to Montague (1970a,b, 1973, 1974). Montague translated a toy fragment of English into a logical representation (Intentional Logic, or IL). This representation was then interpreted in set theory. Semanticists and others

\(^1\)According to Partee (2001), the rigorous application of the method of fragments, as practised by Montague, has fallen out of favour outside of computational linguistics.
often view the set theory as providing the “real” semantics in this Montagovian framework.²

2.1.2. Intensionality with Possible-worlds

Montague sought to demonstrate how intensionality, along with modality, can be analysed using possible worlds. Such possible worlds are given a set-theoretic characterisation, where a possible world is a (consistent) set of propositions (which are true in that world)—or alternatively, a proposition is a set of worlds (in which that proposition is true). This allows for propositions that have the same truth value in the current world to be distinguished if their truth values vary at other worlds. This kind of interpretation can be exploited to model the epistemic modalities. The common view appears to be that the proper analysis of intentionality, and modality, should ultimately be founded upon a set-theoretic characterisation, and it is that characterisation that provides the “real” semantics.

2.1.3. Plurals

A comprehensive semantic theory needs to be able to formalise talk about pluralities of individuals. One approach is to use set theory for plural entities (for example, Landman, 1987), or perhaps some form of lattice theory with a set-theoretic model. Plural nouns then refer to set of entities, and nominal conjunction can be interpreted as set union.

“boys” → \{a, b, c\}

²It is unclear whether it was Montague’s intention that meaning to be reduced to a purely set theoretic characterisation (see Montague, 1969), it does appear to be how his methodology is usually interpreted. For example, Partee (2001) describes the interpretation via IL as “indirect”, indicating the assumed primacy of the set-theoretic interpretation. Dowty et al. (1981, pp263–264) observe that the translations are set up so that the logical representation could be eliminated. Indeed, Intensional Logic as such does not appear to feature in the reprise of Montague’s analysis given by Thomason (1974). In that account, the only thing that seems to intervene between a language and its set-theoretic interpretation is a disambiguated version of the initial language. Thomason himself went on to develop a highly intensional notion of proposition (Thomason, 1980).
“John and Mary” → \{j\} \cup \{m\}

On this account, the arguments of basic predicates are taken to be sets of entities, where singular entities are identified with singleton sets.

2.1.4. Questions and Answers

Although traditionally used for indicatives, set-theoretic possible worlds have been proposed for other kinds of utterances and sentential forms, such as questions and their answers. Following Groenendijk & Stokhof (1984, 1997), questions can be taken to be represented by a partition of worlds. Or to be more specific, exhaustive answers are considered to be partitions of worlds, and the meaning of a question is reduced to the set of all its possible exhaustive answers. For example, a yes–no question partitions the world into two sets. Each set in the partition corresponds to a different possible answer. An answer indicates a partition. A correct answer indicates in which partition the current world is located.

2.1.5. Events

Tense operators (such as future and past) can be construed as relationships between possible worlds. Furthermore, events and instances can be characterised in terms of intervals defined over sets of times (see Kamp, 1971, for example).

2.1.6. Counterfactuals

Set-theoretic possible-worlds interpretations have been proposed for counterfactuals such as “If that kangaroo had no tail it would fall over.” (Lewis, 1973). The idea is that a conditional claim with a counterfactual antecedent is true if the consequent is true in the “nearest” world(s) in which the antecedent is true. The notion of nearest is an ordering relation between worlds where worlds that have minimal plausible differences are closer than those with more dramatic differences. So a world in which a particular kangaroo has no tail is closer than one in which no kangaroo has a tail, or in which kangaroos go around with crutches.
2.2. Why set theory?

Typically there is little overt reflection on the methodology, but as evident in common practice, and its justification, set theory has a *de facto* and *de jure* foundational status that is not bestowed on other formalisms and notation, including IL. We may wonder why set-theoretic interpretations are given primacy over other kinds of interpretations. For example, the intermediate language IL is an intensional version of simple type theory (STT). Here the ontological primitives are types, sets[^3] and possible worlds. If we are concerned with ontological questions, then we may wonder what makes IL less ontological explanatory than pure set theory to the extent that we would wish to eliminate it?[^4]

That set theory is given a special status is perhaps due to familiarity, its formal power, and the foundational role that it plays within mathematics. For some, such semantic theories may even be considered to have “explanatory” or “predictive” power if a mapping can found into expressions of set-theory that have the appropriate behaviour by virtue of the rules of set-theory. More generally, set theory may be seen as playing a foundational role in semantics, and not just mathematics, which sets it apart from other kinds of formalism (and notation).

The apparent foundational status accorded to set theory may have some of its roots in the work of Quine and the earlier Montague, among others. But it should be noted that in the case of Montague’s semantics, the view that IL itself plays no proper role in the analysis of meaning does not appear to have been espoused in Montague’s published work. Indeed, when justifying his adoption of a logical representation language, Montague (1969) seems to make it clear that he thinks something beyond set theory is required when considering some aspects of meaning. This is also noted by Partee (2001), where she says that Montague came to take the view that IL plays a role in the perspicuity of the analysis.

The case could be made that set theory plays an explanatory role for theories of meaning as it allows theoretical notions to be expressed in a familiar language; set theory provides a “vernacular” notation that is

[^3]: It is worth noting that the sets of STT are different from those of Zermelo-Fraenkel set theory, as used in model-theoretic interpretations.

[^4]: Certainly Montague (1969) could be interpreted as endorsing the view that IL should be seen as playing a role that goes beyond its set-theoretic interpretation.
understood by most people who have some mathematical training. In some sense, a set-theoretic characterisation of a semantic theory allows the semantic theory to be presented without any need to justify or explain the language in which it is formalised. But even if we concede this point, set-theoretic interpretations are not without problems.

2.3. Specific Issues

Arguments can be made that reducing everything to set theory has unintended consequences. In the case of intensionality, the set-theoretic possible-worlds interpretation as sets gives the wrong results (Bealer, 1982; Chierchia & Turner, 1988; Fox, 2000; Fox & Lappin, 2005). Furthermore, even if we accept the possible worlds analysis, or some modification of it, there is an issue in that their set-theoretic analysis effectively eliminates them as a distinct ontological category; there are just sets like any other set. Their interpretation as possible worlds (rather than something else) effectively exists as an adjunct to the theory—a layer of explanation that sits outside the set-theoretic semantics itself. This applies to all ontological notions apart from that of “set” itself.

If we assume a straightforward set-theoretic analysis of plurals, then we may wonder what $\{j\}, \{m\}$ represents. It could be considered for controlling distributive inferences (Landman, 1987). But this has been shown to be inappropriate (Schwarzschild, 1992); it merely introduces an apparent ambiguity that serves no independent role.

For questions and answers, the set-theoretic possible worlds model proposed by Groenendijk & Stokhof (1984, 1997) has some apparent oddities. As noted by Koenig (2004), we may wonder why questions should be reduced to sets of exhaustive answers, rather than, for example, sets of pragmatic answers. There also appear to be cases where the most informative answer is one that is expressed in terms of the properties that something has, rather than by giving an extensionally exhaustive list of things that contingently have the right properties in question (Burhans, 1997), which in general may not even be possible.
2.4. Arguments from Benacerraf

It is instructive to reflect on arguments made by Benacerraf (1965). Number theory can be derived using an appropriate “encoding” of numbers as set, such as the one proposed by Zermelo:

\[ 1 = \{\{\}\}, \quad 2 = \{\{\}, \{\}\}, \ldots \]

There are however alternative encodings available, such as von Neumann’s:

\[ 1 = \{\}\}, \quad 2 = \{\{\}, \{\}\}, \ldots \]

These encodings may vary in their behaviour, both from each other, and from the common understanding of what numbers are. If numbers really are sets, we may ask whether it is the case that \(2 \in 3\). The problem here is that questions which ought to be meaningless become meaningful. Furthermore, the answer to such questions is then dependent on the precise nature of the chosen encoding, rather than being a consequence of the theory of numbers as such.

These issues are used by Benacerraf to justify the view that numbers do not refer to some specific concrete realisation, but instead are structural things in themselves. Such structures may be manifest in many other systems, but such systems behave in ways that go beyond the behaviour of numbers. Perhaps similar arguments can be made about formal theories of meaning.

2.5. Arguments from Dummett and Feferman

Jumping straight to set-theory leads to metaphysical questions about meaning and language being overlooked. Such a move presupposes that the ontology of language is that of sets. All other metaphysical options and ontological choices are ignored.

Avoiding a straight reduction to set theory allows ontological issues to be treated more seriously. This is not to say that ontological questions should be decided by purely linguistic criteria, nor that they can be resolved by mere reflection. Rather, our semantic theory can reflect and
inform the answer to the question what does a speaker who knows a language know? (Dummett, 1991).^5

This is perhaps related to Feferman (1992)’s notions of adequacy and faithfulness. It can be argued that a theory of linguistic meaning should not merely record or predict the expected patterns of behaviour. While that may be adequate, it is better if such a theory attempts to be faithful to what we might consider to be the intended interpretation.^6

3. Axiomatic Alternatives

An alternative to a set-theoretic approach of mapping phenomena more-or-less directly into set-theoretic constructs is to allow the adoption of “new” primitives and ontological categories (see Asher, 1993; Peterson, 1997, for example), and devise logical rules and axioms that capture the appropriate inferential behaviour (as in Turner, 1992; Fox, 2000; Fox & Lappin, 2005, for example) in terms of those primitives.

Even if we don’t think meaning (of language) should inform our metaphysics, we should at least allow ontological considerations to inform our analysis of meaning. While a set-theoretic model, or interpretation, can be used to help demonstrate the consistency of an axiomatic account, it should not necessarily be considered to be the primary objective if we wish to take ontological considerations seriously.

3.1. Some examples

Here we look again at the examples of §2.1.

3.1.1. Alternatives to Montague Semantics

There are a number of theories that consider “independent” formalisations of behaviour including Property Theory (Chierchia & Turner, 1988; Bealer, 1982, for example) and Situation Theory (see Barwise, 1989, for

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^5 It seems these particular arguments are independent of Dummett’s case for constructivism.

^6 The notion of faithfulness is arguably related to that of linguistic competence (Chomsky, 1965, p3). Both notions are concerned with issues relating to abstract judgements about a language, not merely its physical manifestation.
example), and *Property Theory with Curry Typing* (Fox & Lappin, 2005).\(^7\) By not reducing everything to set theory, such accounts avoid the problems that stem from the extensional behaviour of sets.

Even with Montague’s account, it should be noted that he appeared to move away from the view that it is acceptable to reduce everything to set theory (Montague, 1969). This seems to suggest he attached some importance to at least some of the ontological notions that exist in IL but not in set theory. The set-theoretic interpretation is then not the intended interpretation, but a model that demonstrates the coherence of the logic.\(^8\)

### 3.1.2. Plurals

To interpret plurals, appropriate structural axioms can be formulated, akin to lattice theory, or merology (Link, 1982; Schwarzschild, 1992, and others). These can be axiomatised independently of any particular set-theoretic interpretation. Indeed, it is then possible to remain neutral on various issues, such as whether full second-order completeness is required for plural definite descriptors, and whether mass terms need to have atomic extensions, and whether or not collective nouns need refer to their constituent members (see Fox, 1998, 2000, for example). Particular set-theoretic structures may *exemplify* such lattice-like behaviour, but that does not mean that we can have no independent characterisation of the appropriate structures.\(^9\)

### 3.1.3. Questions and Answers

There are alternatives to Jeroen Groenendijk and Martin Stokhof formalisation of questions and answers (Ginzburg & Sag, 2000, for example) that do not rely on an overt reduction to possible worlds, but essentially treat questions as a basic ontological category, just as propositions and propositions are treated as basic notions in Property Theory (Chierchia & Turner, 1988; Bealer, 1982).

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\(^7\)Proof-theoretic NL semantics may also count as a general alternative, (Francez & Dyckhoff, 2010, for example).

\(^8\)Indeed, the set-theoretic approach to semantics can be seen to have its modern origins in model theory.

\(^9\)An additional, separate, methodological issue arises in that mathematicians may use the language of set theory to formulate the notion of a lattice.
3.1.4. Counterfactuals

Not all accounts of counterfactuals rely on a possible worlds analysis (Turner, 1981). More generally, it could be argued that some of the key insights that are relied upon even in a set-theoretic analysis do not actually reside within the set-theoretic formulation itself. A case in point is the notion of “closeness”; the insights about this notion and how it should behave are not captured in the standard formal analysis, but they are crucial to the theory.

3.1.5. Events

Similar arguments can be made about set-theoretic characterisations of events. Essentially they can be considered to implement some basic insights, but surely it is those insights, and the ontology that they presuppose, which constitute the “real” theory. When it comes to formalising such notions in their own terms, an axiomatic approach would appear to be a far more appropriate vehicle than any reduction to set theory. Indeed, as Kamp (1971) notes, many may find it unnatural and counterintuitive to characterise events that have duration to be founded on sequences of durationless time points. Even though a set-theoretic interpretation may capture the appropriate behaviour, the ontology can appear to be unnatural. This may be sufficient reason to seek an alternative characterisation. If this is accepted, it suggests that the criteria for an appropriate interpretation of meaning goes beyond mere truth-conditional behaviour within some set-theoretic model, but should also include ontological considerations.

3.2. Summary

The argument is not that adopting an axiomatic approach provides the best or the most comprehensive analysis of the phenomena in question, but that it can help avoid a lack of faithfulness (cf. Feferman, 1992) and allows ontological and other conceptual issues to be treated as first-class concerns rather than some theory-external commentary that plays no substantive role in the final interpretation. Any flaws in the adequacy of an axiomatic account are no different in kind from those of set-theoretic accounts. In brief, whether we use set theory or not, we cannot avoid the hard problems
in semantics, but if we reduce everything to set theory, that may lead to a failure in faithfulness.

3.3. Issues with justification of set theory

We may question why set theory appears to be attributed with a predictive or explanatory role in the literature. Indeed, given that full Zermelo–Fraenkel set theory is such a powerful theory, we may wonder what explanatory power there is in showing that there is a mapping into it, other than the use of common, well-understood notation (§2.2). There is a constructive element here: a mapping from language into set theory has to be provided. But it could be argued that the mapping itself is (merely) a proxy description. The relevant intended behaviour is not explicit in the set-theory by itself. Any formalisation in set theory presupposes a clear understanding of the phenomena and the desired behaviour of the set-theoretic interpretation. We can argue that it is this step corresponds to the axiomatic formalisation of behaviour, and this step in which the real analysis occurs. The interpretation in set theory does not add any conceptual understanding about the notions being formalised. It effectively excludes ontological and metaphysical questions from the formalisation itself, even though such issues that could be considered an essential component of a semantic theory (cf. Dummett, 1991). That being said, the process of creating a set-theoretic interpretation may improve the rigour of the analysis and understanding.

4. Conclusion

We argue that the role of formal theory is to provide an adequate and faithful presentation of observed behaviour. Reductive set-theoretic analyses of semantics are open to a version of Benacerraf’s and Dummett’s criticisms (§2.4 & §2.5). Reductions to set theory perhaps make it too easy to avoid, or fail to take account of, questions of ontology and formal power. Any remaining claims about the inadequacies of “descriptivist” axiomatic accounts compared to set-theoretic reductions must rely on criteria and assumptions that lie outside the domain of formal semantics as such.
The unintended consequences of a set-theoretic analysis, and the potential neglect of ontological matters, do not necessarily mean that it is wrong or inappropriate to use set theory when formalising an analysis of meaning, but rather that it may be inappropriate to consider such analysis in itself as necessarily providing a canonical account, or an appropriate ontology. Indeed, the insights, and conceptual categories used in developing a set-theoretic interpretation, and the descriptive text that provides the intended interpretation, should themselves all be considered part of the semantic analysis. If we wish to internalise these behavioural and ontological insights into our formal analysis, then it would be more appropriate to adopt an axiomatic approach.

References


Fox, Chris (2000), *The Ontology of Language*, CSLI, Stanford.


Lewis, David (1973), Counterfactuals, Harvard University Press.


